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SCATTERING OF ELASTIC WAVES IN DISPERSED COMPOSITES

Sergey V.KUZNETSOV[†] and Irini DJERAN-MAIGRE[‡]

[†]Institute for Problems in Mechanics
Prosp. Vernadskogo 101, Moscow 119526, Russia
e-mail: kuzn-sergey@yandex.ru

[‡]INSA de Lyon, URGC
34 Avenue des Arts, 69621 Villeurbanne Cedex, France
e-mail: Irini.Djeran-Maigre@insa-lyon.fr

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Abstract

A deterministic model is developed for analyzing the scattering cross-sections, speed and energy variation of the plane elastic waves propagating in porous media or dispersed composites. The model is based on the two-scale asymptotic analysis combined with the periodic boundary integral equation method.

1 Introduction

Herein we develop a theoretical model for analyzing energy variation and scattering cross-sections for elastic waves propagating in a heterogeneous medium containing the dispersed inclusions or voids. The considered porous medium or dispersed composite is modeled by a deterministic scheme based on a regular space lattice with inclusions located at the corresponding nodes. Media with inclusions can have different kinds of lattices, each having inclusions of specific geometry and orientation placed at the corresponding nodes.

It is assumed that both the medium and the inclusions are elastic and anisotropic, and that no restrictions on the specific kind of anisotropy is imposed. The other assumption concerns the displacement fields which are supposed to be infinitesimal, so equations of the linear theory of elasticity can be applied.

The main problem for a medium with uniformly distributed inclusions is in its effective characteristic determination; in the case of elasticity it means determination of the effective (or averaged) Young's and shear moduli, Poisson's ratios etc. Along with this main problem several others can be solved in parallel, namely (i) determining level of the microstructural stresses in matrix material, these are highly oscillating stresses which may have high magnitude and can initiate volume fracture; (ii) determining scattering cross sections by inclusions, this is related to the ratio of the energy scattered by these inclusions to the energy of the incident wave. The latter problem is interesting due its direct connection to the non-destructive testing of heterogeneous materials with the discrete inhomogeneities.

The closest solutions in mechanics of heterogeneous media with inclusions can be obtained by application of the two-scale asymptotic analysis [1-3]. In this method it is assumed that two fields exist: (i) the global field which is described by "slow" variables; and, (ii) a local field, having high oscillations, which is described by "fast" variables. Application of the two-scale asymptotic analysis to the problem stated above will be considered in more detail later on.

In the two-scale asymptotic method the effective elasticity tensor is generally represented by

$$\mathbf{C}_0 = \sum_{p=1}^{N} f_p \mathbf{C}_p + \mathbf{K}, \quad \sum_{p=1}^{N} f_p = 1, \tag{1.1}$$

where $\mathbf{C}_0$ is the effective elasticity tensor, f_p is the volume fracture of the *p*-th component, $\mathbf{C}_p$ is the elasticity tensor of the p-th component, N is the total number of different components of the heterogeneous medium, and $\mathbf{K}$ is a correcting tensor, or "corrector". It is clear from (1.1), that the main difficulty in determination of the effective elasticity tensor is in finding the corrector.

Remark. It is interesting to note that Eq. (1.1) covers almost all the existing methods of homogenization by choosing different expressions for the corrector:

a) Thus, if $\mathbf{K} = 0$ the well known Voigt's homogenization is obtained.

b) Taking

$$\mathbf{K} = \sum_{p=1}^{N} - f_p \mathbf{C}_p + \left(f_p \mathbf{C}_p{}^{-1} \right)^{-1} \tag{1.2}$$

and assuming that for any *p* tensor $\mathbf{C}_p$ is invertible along with $\left(f_p \mathbf{C}_p{}^{-1} \right)$, the Reuss homogenization for the elasticity tensor comes out (assumption that $\mathbf{C}_p$ is invertible for any p is not valid for media with pores; in this case the Reuss homogenization produces wrong values for the homogenized elasticity tensor).

Determination of the corrector in the two-scale asymptotic method demands the solution of the cell problem, which in turn consists of (i) setting up a boundary-value problem on the internal boundaries between inclusion(s) and the matrix material in a cell; and, (ii) formulating a periodic boundary-value problem on the outer boundary of a cell. The latter one is non-classical in the sense that it is formulated on the boundary which, due to periodicity, must have angular points and edges.

Along with FEM and finite differences methods, the following other methods for obtaining the solution to the cell problem are known. In [4 - 6], methods based on the Eshelby's transformation strain were applied to analyses of isotropic media with ellipsoidal inclusions. The advantage of these methods resides in their principle possibility to analyze media with anisotropic components, while from the computational point of view these are not very convenient since they lead to the three-dimensional integral equations with weakly singular kernels.

In [7, 8], media with isotropic components were studied by applying a method based on the periodic fundamental solution for isotropic medium, which originally was constructed in [9]. Because of multipolar expansions used for the solution of the inner boundary value problem this method is confined to inclusions of spherical form. A similar approach was also used in the case of isotropic composites, but it was based on the Galerkin technique for solution of the inner boundary value problem [10].

Periodic fundamental solutions for media with arbitrary anisotropy were developed in [11], In combination with the Boundary Integral Equation Method (BIEM) these fundamental solutions were applied to the cell problem for composites with anisotropic inhomogeneities and porous media in [12, 13], analysis of microstructural stresses in the matrix material was considered in [14]. Problems of wave scattering by pores were studied in [15] by application of the same method. Some of obvious advantages of this method are due to potential possibility to reduce the solution of the inner boundary-value problem to a summation of the rapidly convergent series, while periodic boundary conditions on the outer boundary are satisfied automatically due to periodicity of the fundamental solution.

Scattering of elastic waves in the dispersed composites and porous media are generally studied at the long wave assumption [15-20], this means that the wave length surpasses considerably the lattice period. Some an additional assumption of the constancy of the wave speed in a lattice cell is made [21, 22], this is known as the Rayleigh approximation. Non-linear problems related to the analysis of wave scattering by the inclusions or pores are treated in [23-25].

The following analysis is targeted to obtaining scattering cross sections for the plane harmonic wave scattered by periodically distributed inclusions or voids in an anisotropic composite with the arbitrary elastic anisotropy on the bases of the two-scale asymptotic analysis; see also [28].

2 Basic notations

A homogeneous elastic anisotropic medium is considered. The equations of equilibrium can be written in the form:

$$\mathbf{A}(\partial_x)\mathbf{u} = -\operatorname{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u} = 0, \tag{2.1}$$

where $\mathbf{u}$ is a displacement field. It is assumed that the tensor of elasticity satisfies the condition of positive definiteness, which is generally adopted for problems of mechanics.

Applying the Fourier transform

$$f^\wedge(\xi) = \int f(\mathbf{x})\exp(2\pi i\mathbf{x}\cdot\xi)\,dx, \quad \xi \in R^3 \tag{2.2}$$

to Eqs. (2.1) gives the following symbol of the operator $\mathbf{A}$:

$$\mathbf{A}^\wedge(\xi) = (2\pi)^2\,\xi\cdot\mathbf{C}\cdot\xi. \tag{2.3}$$

From the definition of the fundamental solution $\mathbf{E}$, the following formula for the corresponding symbol can be obtained:

$$\mathbf{E}^\wedge(\xi) = \mathbf{A}^\wedge(\xi)^{-1}. \tag{2.4}$$

The expression (2.4) shows that symbol $\mathbf{E}^\wedge$ is also strongly elliptic, positively homogeneous of degree -2 with respect to $|\xi|$, and analytical everywhere in $R^3 \setminus 0$.

Remark. The Fourier inversion of the expression (2.4) and procedures for construction of the fundamental solution, are discussed in [26, 27].

3 Spatially periodic fundamental solution

Consider a homogeneous anisotropic medium, loaded by periodically distributed force singularities, located in nodes $\mathbf{m}$ of a spatial lattice Λ.

Let $\mathbf{a}_i$, $(i = 1,2,3)$ be linearly independent vectors of the main periods of the lattice, so that each of the nodes can be represented in the form:

$$\mathbf{m} = \sum_i m_i \mathbf{a}_i \, , \tag{3.1}$$

where $m_i \in Z$ are the integer-valued coordinates of the node $\mathbf{m}$ in the basis $(\mathbf{a}_i)$. The adjoint basis $(\mathbf{a}_i{*})$ is introduced in such a manner that $\mathbf{a}_i{*} \cdot \mathbf{m} = m_i$. The lattice corresponding to the adjoint basis is denoted by $\Lambda{*}$.

Now, periodic delta-function corresponding to the singularities disposed at the nodes of the lattice Λ has the form:

$$\delta_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}{*}\in\Lambda{*}} \exp(-2\pi i \mathbf{x} \cdot \mathbf{m}{*}) \, , \tag{3.2}$$

where V_Q is the volume of the fundamental region (cell) Q. Expression (3.2) defines the periodic delta-function uniquely.

Substitution of the periodic fundamental solution $\mathbf{E}_p$ in Eq. (2.1) yields

$$\mathbf{A}(\partial_{\mathbf{x}})\mathbf{E}_p(\mathbf{x}) = \delta_p(\mathbf{x})\mathbf{I} \, , \tag{3.3}$$

where $\mathbf{I}$ is the identity matrix. Looking for $\mathbf{E}_p$ also in the form of harmonic series and taking into account representation (3.2), it is possible to get:

$$\mathbf{E}_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}{*}\in\Lambda_0{*}} \mathbf{E}^\wedge(\mathbf{m}{*}) \exp(-2\pi i \mathbf{x} \cdot \mathbf{m}{*}) \, , \tag{3.4}$$

where $\Lambda_0{*}$ is the adjoint lattice without the zero node. It should be noted that expression (3.4) defines a periodic fundamental solution up to an additive (tensorial) constant.

Lemma 1. The series on the right side of Eq. (3.4) is convergent in the L^1-topology, defining the fundamental solution of the class $\overline{L^1}(Q, R^3 \otimes R^3)$, where $\overline{L^1}$ is a class of integrable in Q functions with the zero mean value.

Proof of the lemma can be found in [11].

4 Effective elasticity tensor and scattering cross section

For clarity and simplicity it will be assumed that the considered medium has only one kind of uniformly distributed inhomogeneities placed in nodes of spatial lattice Λ. The region occupied by an individual inhomogeneity in a cell Q is denoted by Ω.

The two-scale asymptotic analyses being applied to such a medium produces the following expression for the corrector [12]:

$$\mathbf{K} = -V_Q^{-1} \int_{\partial\Omega} \mathbf{C} \cdot\cdot (\nu_{\mathbf{Y}} \otimes \mathbf{H}(\mathbf{Y}))\, dY \,, \tag{4.1}$$

where $\mathbf{Y}$ are the "fast" variables, $\mathbf{H}$ is the third-order tensorial field, being a solution of the following boundary value problem:

$$\begin{aligned} &\mathbf{A}(\partial_Y)\mathbf{H}(\mathbf{Y}) = 0, \;\; \mathbf{Y} \in Q \setminus \Omega \\ &\mathbf{T}(\nu_Y, \partial_Y)\mathbf{H}(\mathbf{Y}) \big|_{\partial\Omega} = -\nu_Y \cdot \mathbf{C} \end{aligned} \;. \tag{4.2}$$

In Eqs. (4.1) and (4.2) ν_Y represents the field of external unit normal to the boundary $\partial\Omega$, and the elasticity tensor Ω is defined by:

$$\mathbf{C} = \mathbf{C}_2 - \mathbf{C}_1, \tag{4.3}$$

where $\mathbf{C}_2$ is referred to the matrix material, and $\mathbf{C}_1$ to inclusions. Strong ellipticity of the tensor $\mathbf{C}$ is also assumed.

Lemma 2. Under assumptions stated above, boundary-value problem (3.9) admits the unique solution.

Proof of the lemma can be found in [11, 12].

Remark. Supposition that the tensor **C** in the left-hand side of Eq. (4.3) is not strong elliptic, violates proof of Lemma 2.

Now, the solution of the boundary value problem (4.2) for the traction field can be constructed by applying boundary integral equation method, giving the following representation for the desired solution [12]:

$$\left(\tfrac{1}{2}\mathbf{I} + \mathbf{S}\right)\mathbf{H}(\mathbf{Y}') = \mathbf{H}_c \qquad \mathbf{Y}' \in \partial\Omega, \tag{4.4}$$

where $\mathbf{H}_c$ is a constant tensor, and $\mathbf{S}$ is a singular integral operator resulting from a restriction of the double-layer potential on the surface $\partial\Omega$. Some of the relevant properties of operator $\mathbf{S}$ are discussed in [13].

Substitution of Eq. (3.4) for periodic fundamental solutions in the expression for the operator S allows to obtain a lower (on energy) bound for the corrector; i.e.

$$\mathbf{K}_l = -8\pi^2 V_Q^{-2} \sum_{\mathbf{m}^* \in \Lambda_0^*} \left(\chi^\wedge_\Omega(\mathbf{m}^*)\right)^2 \mathbf{C}\cdot\cdot\,\mathbf{m}^* \otimes \mathbf{E}^\wedge(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot\,\mathbf{C}, \tag{4.5}$$

where $\chi^\wedge_\Omega$ is the Fourier image of the characteristic function of the region Ω. An expression for the upper bound can be obtained similarly [12, 13].

Theorem. Series appearing on the right side of Eq. (4.5) is absolutely convergent, provided Ω is a proper open region in Q .

Proof of the theorem can be found in [12, 13]

Remark. Proof of convergence of the series analogous to (4.5) for very thin inclusions or cracks, is to be studied separately, as in this case a special asymptotic analysis is needed.

As was shown in [13, 14], the energy level W_{osc} of the microstructural highly oscillating stresses for the case of porous medium is defined by:

$$W_{osc} = \tfrac{1}{2}\varepsilon_0 \cdot\cdot \mathbf{K} \cdot\cdot \varepsilon_0 , \tag{4.6}$$

where ε_0 represents the uniform deformation field, and $\mathbf{K}$ is the corrector obtained by Eq. (4.5).

Similarly, having applied terminology used in quantum mechanics, the scattering cross-section S for the porous medium can be obtained by the following expression [15]:

$$S = (1 - f)^{-1} \left| \frac{\varepsilon_0 \cdot\cdot \mathbf{K} \cdot\cdot \varepsilon_0}{\varepsilon_0 \cdot\cdot \mathbf{C} \cdot\cdot \varepsilon_0} \right| , \tag{4.7}$$

where f is the porous ratio and $\mathbf{C}$ is the elasticity tensor for the matrix material, in expression (4.7) the homogeneous deformation field ε_0 corresponds to the amplitude deformation on the wave front:

$$\varepsilon_0 = \frac{1}{2}\left(\mathbf{n} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{n}\right), \tag{4.8}$$

In (4.8) $\mathbf{a}$ is the polarization vector of the bulk wave and $\mathbf{n}$ is the unit vector normal to the plane wave front. Polarization vector $\mathbf{a}$ in the right-hand side of (4.8) should satisfy the propagation condition

$$\left(\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n}\right) \cdot \mathbf{a} = \rho c^2 \mathbf{a}, \tag{4.9}$$

where c is the speed of the corresponding bulk wave, and ρ is the density.

Remark. In [12-15] some examples for the corrector obtained by Eq. (4.5), and corresponding to inclusions or voids of some canonical shapes, are presented.

As is seen from Eq. (4.7), the scattering cross-section heavily depends upon the corrector $\mathbf{K}$ (and the applied homogenization technique). For example, Voigt's homogenization (see Remark in Sec. 1) necessary leads to absence of any scattering irrespective of nature of a dispersed composite or porous media, while Reuss homogenization leads to infinite scattering cross-section for any porous medium. This underlines the fact of necessity to chose the closest technique for evaluating the corrector.

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